

Large losses - probability minimizing approach

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Abstract

The probability minimizing problem of large losses of portfolio in discrete and continuous time models is studied. This gives a generalization of quantile hedging presented in [3].

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1 Introduction

Let (S_t) be a d -dimensional semimartingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ which represents the stock prices. We denote by \mathcal{Q} the set of all martingale measures, it means that $Q \in \mathcal{Q}$ if $Q \sim P$ and (S_t) is a martingale with respect to Q . Let H be a \mathcal{F} measurable random variable called contingent claim. It is known that on such market we have two prices: the buyer's price $u_b = \inf_{Q \in \mathcal{Q}} \mathbf{E}^Q[H]$ and the seller's price $u_s = \sup_{Q \in \mathcal{Q}} \mathbf{E}^Q[H]$, which usually are different. A natural question arises : what price from the so called arbitrage-free interval $[u_b, u_s]$ should be chosen? This problem was a motivation for introducing risk measures on financial markets. Various approaches were presented to answer this question, see for instance [1],[2],[3],[4],[6].

In [3] Föllmer, and Leukert study the quantile hedging problem. They define a random variable $\varphi_{x,\pi}$ connected with the strategy (x, π) by:

$$\varphi_{x,\pi} = \mathbf{1}_{\{X_T^{x,\pi} \geq H\}} + \frac{X_T^{x,\pi}}{H} \mathbf{1}_{\{X_T^{x,\pi} < H\}},$$

where $X_T^{x,\pi}$ is the terminal value of the portfolio connected with the strategy π starting from the initial endowment x . If $x \geq u_s$ then for the hedging strategy $\tilde{\pi}$ we have $\mathbf{E}[\varphi_{x,\tilde{\pi}}] = 1$, otherwise $\mathbf{E}[\varphi_{x,\pi}] < 1$ for each π . The aim of the trader is to maximize $\mathbf{E}[\varphi_{x,\pi}]$ over π from the set of all admissible strategies. Actually, the motivation of quantile hedging was a slightly different problem, namely

$$P(X_T^{x,\pi} \geq H) \longrightarrow \max_{\pi}.$$

This problem was solved by the above approach only in a particular case.

Now assume that investor has a loss function $u : [0, \infty) \longrightarrow [0, \infty)$, $u(0) = 0$, which is assumed to be continuous and strictly increasing, and he accepts small losses of the portfolio. It

means he has no objections to losses s.t. $u((H - X_T^{x,\pi})^+) \leq \alpha$, where $\alpha \geq 0$ is a level of acceptable losses fixed by the investor. He wants to avoid losses which exceed α . As the optimality criterion we admit maximizing probability that losses are small. More precisely, the problem is

$$P[u((H - X_T^{x,\pi})^+) \leq \alpha] \longrightarrow \max_{\pi},$$

where π is an admissible strategy. Notice that for $\alpha = 0$ we obtain an original problem of quantile hedging.

The paper is organized as follows. In section 2 we precisely formulate the problem. It turns out that the solution on complete markets has a clear economic interpretation. It is presented in section 3. Sections 4 and 5 provide examples of Black-Scholes model and the CRR model. For B-S model explicit solution is found while for the CRR model existence is clear, but solutions are found for some particular cases. In section 6 result for incomplete markets is proved and presented in a one step trinomial model.

2 Problem formulation

We consider financial markets with either discrete or continuous time and with finite horizon T . Let S_t be a d dimensional semimartingale describing evolution of stocks' prices on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. $X_t^{x,\pi}$ is a wealth process connected with a pair (x, π) , where π is a predictable process describing self-financing strategy and x is an initial endowment. Thus the wealth process is defined by: $X_0^{x,\pi} = x$, $X_t^{x,\pi} = \pi_t \cdot S_t$ and the self-financing condition means that

$$\begin{aligned} \pi_t \cdot S_t &= \pi_{t+1} \cdot S_t && \text{in case of discrete time model} \\ dX_t^{x,\pi} &= \pi_t dS_t, \quad \pi \in L(S) && \text{in case of continuous time model; } L(S) \text{ is the set of predictable} \\ &&& \text{processes integrable w.r. to } S. \end{aligned}$$

For simplicity assume that the interest rate is equal to zero and that the set of all martingale measures \mathcal{Q} , so that measures Q that S_t is a martingale with respect to Q and $Q \sim P$, is not empty. Among all self-financing strategies we distinguish set \mathcal{A} of all admissible strategies which satisfy two additional conditions: $X_t^{x,\pi} \geq 0$ for all t and $X_t^{x,\pi}$ is a supermartingale with respect to each $Q \in \mathcal{Q}$. If $X_t^{x,\pi} \geq 0$ then the second requirement is automatically satisfied for S being a continuous semimartingale, since then the wealth process is a Q -local martingale bounded from below, so by Fatou's lemma it is a supermartingale. In discrete time $X_t^{x,\pi}$ is even a martingale, see [5] Th. 2. Let H be a nonnegative, \mathcal{F}_T measurable random variable, called contingent claim, which satisfies condition $H \in L^1(Q)$ for each $Q \in \mathcal{Q}$. Its price at time 0 is given by $v_0 = \sup_{Q \in \mathcal{Q}} \mathbf{E}^Q[H]$. This means that there exists a strategy $\tilde{\pi} \in \mathcal{A}$ such that $X_T^{v_0, \tilde{\pi}} \geq H$. Such $\tilde{\pi}$ is called a hedging strategy. Now assume that we are given an initial capital $0 \leq x_0 < v_0$. The question arises, what is an optimal strategy for such endowment? As an optimality criterion we admit minimizing probability of a large loss. Let $u : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing, continuous function such that $u(0) = 0$. Such function will be called a loss function. Let $\alpha \geq 0$ be a level of acceptable losses. We are searching for a pair (x, π) such that

$$\begin{aligned} P[u((H - X_T^{x,\pi})^+) \leq \alpha] &\longrightarrow \max_{\pi \in \mathcal{A}}, \\ x &\leq x_0. \end{aligned}$$

If there exists a solution (x, π) of the problem above, then it will be called optimal.

3 Complete models

Let $\mathcal{Q} = \{Q\}$, so the martingale measure is unique. Recall that in this case each nonnegative Q - integrable contingent claim X can be replicated. It means that there exists $\tilde{\pi}$ such that $X_T^{v_0, \tilde{\pi}} = X$, where $v_0 = \mathbf{E}^Q[X]$. In complete case the solution of our problem has a clear economic interpretation. Let us start with the basic theorem describing the solution.

Theorem 3.1 *If there exists $\tilde{X} \in L_+^0$ which is a solution of the problem*

$$P[u((H - X)^+) \leq \alpha] \longrightarrow \max \\ \mathbf{E}^Q[X] \leq x_0$$

then the replicating strategy for \tilde{X} is optimal.

Proof : Recall that for $(x, \pi), \pi \in \mathcal{A}$ the wealth process $X_t^{x, \pi}$ is a supermartingale with respect to Q . Thus we have $\mathbf{E}^Q[X_T^{x, \pi}] \leq x \leq x_0$ and $P[u((H - X_T^{x, \pi})^+) \leq \alpha] \leq P[u((H - \tilde{X})^+) \leq \alpha]$. \square

The main difficulty in this theorem is that we do not have an existence result for \tilde{X} and any method of constructing which could be used for practical applications. However, we show that the problem can be reduced to a simpler one by considering a narrower class of random variables than L_+^0 and for this class in some situations the problem can be explicitly solved. This is an idea of considering strategies of class \mathcal{S} which we explain below.

Economic motivation for introducing strategies of class \mathcal{S}

For $(x, \pi), \pi \in \mathcal{A}$ consider two sets: $A = \{\omega \in \Omega : u((H - X_T^{x, \pi})^+) \leq \alpha\}$ and its compliment A^c . Basing on (x, π) let us build a modified strategy $(\tilde{x}, \tilde{\pi})$ in the following way. On A investor's loss is smaller than α . However, from our point of view it can be as large as possible, but not larger than α . Therefore let $(\tilde{x}, \tilde{\pi})$ be such that on A holds $u((H - X_T^{\tilde{x}, \tilde{\pi}})^+) = \alpha$. On A^c investor did not manage to hedge large loss, so the portfolio value can be as well equal to 0. Such $(\tilde{x}, \tilde{\pi})$ we will regard as a strategy of class \mathcal{S} . What is an advantage of such modification ? It turns out that $\tilde{\pi} \in \mathcal{A}$ and the following inequalities hold:

$$P[u((H - X_T^{\tilde{x}, \tilde{\pi}})^+) \leq \alpha] = P[u((H - X_T^{x, \pi})^+) \leq \alpha], \quad \tilde{x} \leq x.$$

This fact is a motivation for searching the solution of the problem only among strategies of class \mathcal{S} . Below we present this idea in a more precise way.

Definition 3.2 *Random variable $X \in L_+^0$ is of class \mathcal{S} if there exists $A \in \mathcal{F}$ containing $\{u(H) \leq \alpha\}$ such that*

1. *on A we have*

- (a) *if $u(H) \leq \alpha$ then $X = 0$*
- (b) *if $u(H) > \alpha$ then $u(H - X) = \alpha$*

2. *on A^c we have $X = 0$.*

Notice that on the set A we have $X = 0$ if $H \leq u^{-1}(\alpha)$ and $X = H - u^{-1}(\alpha)$ if $H > u^{-1}(\alpha)$. Thus on A we have $X = (H - u^{-1}(\alpha))^+$. Since $X = 0$ on A^c we obtain that $X = \mathbf{1}_A(H - u^{-1}(\alpha))^+$. In other words $X \in \mathcal{S}$ if it is of the form $X = \mathbf{1}_A(H - u^{-1}(\alpha))^+$ for some $A \in \mathcal{F}$ such that $A \supseteq \{u(H) \leq \alpha\}$.

Lemma 3.3 For each $X \in L_+^0$ such that $\mathbf{E}^Q[X] \leq x_0$ there exists a random variable $Z \in \mathcal{S}$ such that $\mathbf{E}^Q[Z] \leq \mathbf{E}^Q[X]$ and

$$P[u((H - X)^+) \leq \alpha] = P[u((H - Z)^+) \leq \alpha].$$

Proof : Let us define $A := \{\omega : u(H - X)^+ \leq \alpha\}$. Then $Z := \mathbf{1}_A(H - u^{-1}(\alpha))^+ \in \mathcal{S}$ and we have

$$\begin{aligned} P[u((H - Z)^+) \leq \alpha] &= P[u((H - \mathbf{1}_A(H - u^{-1}(\alpha))^+)^+) \leq \alpha] \\ &= P[\omega \in A : u((H - (H - u^{-1}(\alpha))^+)^+) \leq \alpha] + P[\omega \in A^c : u(H) \leq \alpha] \\ &= P[\omega \in A \cap \{u(H) \leq \alpha\} : u(H) \leq \alpha] + P[\omega \in A \cap \{u(H) > \alpha\} : u(u^{-1}(\alpha)) \leq \alpha] \\ &= P(A). \end{aligned}$$

On the set A^c holds $Z = 0 \leq X$. On A if $u(H) \leq \alpha$ then $Z = 0 \leq X$ and if $u(H) > \alpha$ then $Z = H - u^{-1}(\alpha) \leq X$. Thus we have $Z \leq X$ and $\mathbf{E}^Q[Z] \leq \mathbf{E}^Q[X]$. \square

Remark 3.4 The above calculations show that for any $X = \mathbf{1}_B(H - u^{-1}(\alpha))^+ \in \mathcal{S}$ holds

$$P(u(H - X)^+ \leq \alpha) = P(B).$$

Using lemma 3.3 and remark 3.4 we can reformulate theorem 3.1 in the following form.

Theorem 3.5 If there exists set $\tilde{A} \supseteq \{u(H) \leq \alpha\}$ which is a solution of the problem :

$$P(A) \longrightarrow \max \tag{3.5.1}$$

$$\mathbf{E}^Q[\mathbf{1}_A(H - u^{-1}(\alpha))^+] \leq x_0 \tag{3.5.2}$$

then the replicating strategy for $\mathbf{1}_{\tilde{A}}(H - u^{-1}(\alpha))^+$ is optimal.

Proof : Indeed, by lemma 3.3 the problem

$$P[u(H - X)^+ \leq \alpha] \longrightarrow \max, \quad \mathbf{E}^Q[X] \leq x_0, \quad X \in L_+^0$$

can be replaced by

$$P[u(H - X)^+ \leq \alpha] \longrightarrow \max, \quad \mathbf{E}^Q[X] \leq x_0, \quad X \in \mathcal{S}.$$

However, by remark 3.4 we know that for $X = \mathbf{1}_A(H - u^{-1}(\alpha))^+ \in \mathcal{S}$ we have $P[u(H - X)^+ \leq \alpha] = P(A)$ and the required formulation is obtained. \square

Remark 3.6 Let us consider the optimizing problem from theorem 3.5 given by 3.5.1 and 3.5.2 but without the requirement that $\tilde{A} \supseteq \{u(H) \leq \alpha\}$. Notice that if $P(u(H) \leq \alpha) > 0$ then the solution \tilde{A} must contain $\{u(H) \leq \alpha\}$. Suppose the contrary and define $\tilde{\tilde{A}} := \tilde{A} \cup \{u(H) \leq \alpha\}$. Then $\mathbf{E}^Q[\mathbf{1}_{\tilde{\tilde{A}}}(H - u^{-1}(\alpha))^+] = \mathbf{E}^Q[\mathbf{1}_{\tilde{A}}(H - u^{-1}(\alpha))^+] \leq x_0$ and $P(\tilde{\tilde{A}}) > P(\tilde{A})$ what is a contradiction. This shows that the requirement $\tilde{A} \supseteq \{u(H) \leq \alpha\}$ in the theorem 3.5 can be dropped.

In some particular cases the existence and construction of the set \tilde{A} can be solved by using Neyman-Pearson lemma. To this end let us introduce a measure \bar{Q} which is absolutely continuous with respect to Q by:

$$\frac{d\bar{Q}}{dQ} = \frac{(H - u^{-1}(\alpha))^+}{\mathbf{E}^Q[(H - u^{-1}(\alpha))^+]}$$

Then set \tilde{A} solves the following problem

$$P(A) \longrightarrow \max$$

$$\bar{Q}(A) \leq \frac{x_0}{\mathbf{E}^Q[(H - u^{-1}(\alpha))^+]}$$

To make the paper self-contained we present a part of the Neyman-Pearson lemma. Let P_1 and P_2 be two probability measures such that there exists density $\frac{dP_1}{dP_2}$.

Lemma 3.7 *If there exists constant β such that $P_2\{\frac{dP_1}{dP_2} \geq \beta\} = \gamma$ then $P_1\{\frac{dP_1}{dP_2} \geq \beta\} \geq P_1(B)$ for any set B satisfying $P_2(B) \leq \gamma$.*

Proof : Let B be a set satisfying $P_2(B) \leq \gamma$ and denote $\tilde{B} := \{\frac{dP_1}{dP_2} \geq \beta\}$. Then we have

$$\begin{aligned} P_1(\tilde{B}) - P(B) &= \int_{\Omega} (\mathbf{1}_{\tilde{B}} - \mathbf{1}_B) dP_1 = \int_{\frac{dP_1}{dP_2} \geq \beta} (\mathbf{1}_{\tilde{B}} - \mathbf{1}_B) dP_1 + \int_{\frac{dP_1}{dP_2} < \beta} (\mathbf{1}_{\tilde{B}} - \mathbf{1}_B) dP_1 \\ &\geq \int_{\frac{dP_1}{dP_2} \geq \beta} (\mathbf{1}_{\tilde{B}} - \mathbf{1}_B) \beta dP_2 - \int_{\frac{dP_1}{dP_2} < \beta} \mathbf{1}_B \beta dP_2 \\ &= \beta \left(\int_{\tilde{B}} dP_2 - \int_B dP_2 \right) = \beta(\gamma - P_2(B)) \geq 0. \end{aligned}$$

□

This lemma is useful for the Black-Scholes model since there the condition $\bar{Q}\{\frac{dP}{dQ} \geq \beta\} = \frac{x_0}{\mathbf{E}^Q[(H - u^{-1}(\alpha))^+]}$ is satisfied. However, in case of discrete Ω this condition no longer holds. This will be shown in the example of the *CRR* model.

4 Black-Scholes model

Here we follow an example presented in [3]. The stock price S_t is given by

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = s,$$

where μ and $\sigma > 0$ are constants and W_t is a standard Brownian motion. For this model

$$S_t = s e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

and the unique martingale measure Q is given by

$$\frac{dQ}{dP} = e^{-\frac{\mu}{\sigma}W_T - \frac{1}{2}(\frac{\mu}{\sigma})^2T}.$$

Moreover, the process $W_t^* = W_t + \frac{\mu}{\sigma}t$ is a Brownian motion with respect to Q . Notice that the density of the martingale measure can be expressed in term of S_T , namely

$$\frac{dQ}{dP} = c S_T^{-\frac{\mu}{\sigma^2}}, \quad \text{where } c \text{ is some constant.}$$

We study a risk minimizing problem for a European call option with strike K . Recall that the problem is reduced to constructing set \tilde{A} being a solution of

$$P(A) \longrightarrow \max$$

$$\bar{Q}(A) \leq \frac{x_0}{\mathbf{E}^Q[(S_T - K - u^{-1}(\alpha))^+]},$$

where measure \bar{Q} is as in the previous section :

$$\frac{d\bar{Q}}{dQ} = \frac{(S_T - K - u^{-1}(\alpha))^+}{\mathbf{E}^Q[(S_T - K - u^{-1}(\alpha))^+]}$$

Notice, that the superscript " + " above can be dropped since for any $a, b, c \geq 0$ holds $((a - b)^+ - c)^+ = (a - b - c)^+$. According to Neyman-Pearson lemma we are searching for the set \tilde{A} of the form:

$$\left\{ \frac{dP}{dQ} \geq c_1 \right\} = \left\{ \frac{dP}{dQ} \geq c_2(S_T - K - u^{-1}(\alpha))^+ \right\} = \left\{ S_T^{\frac{\mu}{\sigma^2}} \geq c \cdot c_2(S_T - K - u^{-1}(\alpha))^+ \right\},$$

where c_1, c_2 are nonnegative constants such that

$$E^Q[\mathbf{1}_{\tilde{A}}(S_T - K - u^{-1}(\alpha))^+] = x_0. \quad (4.0.3)$$

Let us consider two cases.

1) $\mu \leq \sigma^2$

Then the function $x \longrightarrow x^{\frac{\mu}{\sigma^2}}$ is concave and has 0 in 0 and thus the solution is given by

$\tilde{A} = \{S_T \leq c_3\} = \{W_T^* \leq c_4\}$, where c_3 and c_4 s.t. $c_3 = se^{\sigma c_4 - \frac{1}{2}\sigma^2 T}$ are constant numbers satisfying 3.0.3. The optimal strategy is a strategy which replicates the following contingent claim:

$$\begin{aligned} \mathbf{1}_{\tilde{A}}(S_T - K - u^{-1}(\alpha))^+ &= \mathbf{1}_{\{S_T \leq c_3\}}(S_T - K - u^{-1}(\alpha))^+ \\ &= (S_T - K - u^{-1}(\alpha))^+ - (S_T - c_3)^+ - (c_3 - K - u^{-1}(\alpha))\mathbf{1}_{\{S_T > c_3\}} \end{aligned}$$

and the corresponding probability is equal

$$P(\tilde{A}) = P(W_T^* \leq c_4) = \Phi\left(\frac{c_4 - \frac{\mu}{\sigma}T}{\sqrt{T}}\right).$$

For calculating constants c_3 and c_4 from 4.0.3 we use formula for pricing European call option.

$$\begin{aligned} \mathbf{E}^Q\left[(S_T - K - u^{-1}(\alpha))^+ - (S_T - c_3)^+ - (c_3 - K - u^{-1}(\alpha))\mathbf{1}_{\{S_T > c_3\}}\right] &= \\ s\Phi(\bar{d}_+) - (K + u^{-1}(\alpha))\Phi(\bar{d}_-) - s\Phi\left(\frac{-c_4 + \sigma T}{\sqrt{T}}\right) + c_3\Phi\left(-\frac{c_4}{\sqrt{T}}\right) - \\ (c_3 - K - u^{-1}(\alpha))Q\{W_T^* > c_4\} &= \\ s\Phi(\bar{d}_+) - (K + u^{-1}(\alpha))\Phi(\bar{d}_-) - s\Phi\left(\frac{-c_4 + \sigma T}{\sqrt{T}}\right) + (K + u^{-1}(\alpha))\Phi\left(-\frac{c_4}{\sqrt{T}}\right) &= x_0, \end{aligned}$$

where $\bar{d}_{\pm} = -\frac{1}{\sigma\sqrt{T}} \ln\left(\frac{K+u^{-1}(\alpha)}{s}\right) \pm \frac{1}{2}\sigma\sqrt{T}$ and Φ stands for the distribution function of the $N(0,1)$ distribution.

2) $\mu > \sigma^2$

In this case the function $x \rightarrow x^{\frac{\mu}{\sigma^2}}$ is convex and therefore our solution is of the form

$$\tilde{A} = \{S_T < c_5\} \cup \{S_T > c_6\} = \{W_T^* < c_7\} \cup \{W_T^* > c_8\}$$

where $c_5 < c_6$ are two solutions of the equation $x^{\frac{\mu}{\sigma^2}} = \bar{c}(x - K - u^{-1}(\alpha))^+$, where \bar{c} is a constant number s.t. 4.0.3 holds. Constants c_7, c_8 are given by $c_5 = se^{\sigma c_7 - \frac{1}{2}\sigma^2 T}$, $c_6 = se^{\sigma c_8 - \frac{1}{2}\sigma^2 T}$. The optimal strategy is a strategy which replicates the following contingent claim:

$$\begin{aligned} \mathbf{1}_{\tilde{A}}(S_T - K - u^{-1}(\alpha))^+ = \\ (S_T - K - u^{-1}(\alpha))^+ - (S_T - c_5)^+ - (c_5 - K - u^{-1}(\alpha))\mathbf{1}_{\{S_T > c_5\}} + (S_T - c_6)^+ + \\ (c_6 - K - u^{-1}(\alpha))\mathbf{1}_{\{S_T > c_6\}} \end{aligned}$$

and the corresponding probability is equal

$$P(\tilde{A}) = P(W_T^* < c_7) + P(W_T^* > c_8) = \Phi\left(\frac{c_7 - \frac{\mu}{\sigma}T}{\sqrt{T}}\right) + \Phi\left(-\frac{c_8 - \frac{\mu}{\sigma}T}{\sqrt{T}}\right).$$

Now we need to determine all necessary constants. Using the same methods as in the previous case we obtain

$$\begin{aligned} \mathbf{E}^Q[\mathbf{1}_{\tilde{A}}(S_T - K - u^{-1}(\alpha))^+] &= s\Phi(\bar{d}_+) - (K + u^{-1}(\alpha))\Phi(\bar{d}_-) - \\ s\Phi\left(-\frac{c_7}{\sqrt{T}} + \sigma\sqrt{T}\right) &+ s\Phi\left(-\frac{c_8}{\sqrt{T}} + \sigma\sqrt{T}\right) + (K + u^{-1}(\alpha))\left(\Phi\left(-\frac{c_7}{\sqrt{T}}\right) - \Phi\left(-\frac{c_8}{\sqrt{T}}\right)\right) = x_0 \end{aligned} \quad (4.0.4)$$

Summarizing, constants are determined by 4.0.4 and by the fact that c_5, c_6 are solutions of the equation $x^{\frac{\mu}{\sigma^2}} = \bar{c}(x - K - u^{-1}(\alpha))^+$, where \bar{c} is a positive constant.

5 CRR model

Let $(S_n)_{n=0,1,2,\dots,N}$ be a stock price given by

$$S_{n+1} = S_n(1 + \rho_n), \quad S_0 = S,$$

where (ρ_n) is a sequence of independent random variables such that $p := P(\rho_n = u) = 1 - P(\rho_n = d)$, where $u > d, u > 0, d < 0$. This means that at any time the price process S_n can increase to the value $S_n(1 + u)$ or decrease to $S_n(1 + d)$. We assume that $p \in (0, 1)$. It is known, that the unique martingale measure for this model is given by $p^* := \frac{-d}{u-d}$.

Let us study the risk minimizing problem for the call option with strike K . Let us denote $(S_N - K)^+ := (S_N - K - u^{-1}(\alpha))^+$ and consider two measures: the objective one P

$$P(\omega_k) = p^k(1 - p)^{N-k}$$

and the measure \bar{Q} (which is not necessarily a probability measure) given by

$$\bar{Q}(\omega_k) := (S(1+u)^k(1+d)^{N-k} - \bar{K})^+ p^{*k} (1-p^*)^{N-k}.$$

Here ω_k means an elementary event for which the number of jumps upwards is equal to k . Our aim is to find set \tilde{A} which solves:

$$\begin{aligned} P(A) &\longrightarrow \max \\ \bar{Q}(A) &\leq x_0. \end{aligned}$$

For the *CRR* model existence of the required set \tilde{A} is clear since Ω is finite. However we want to find it explicitly. Unfortunately, the Neyman-Pearson lemma for the measures P and \bar{Q} can not be applied here since Ω is discrete and the condition

$$\bar{Q}\left\{\frac{dP}{d\bar{Q}} \geq a(H - u^{-1}(\alpha))^+\right\} = \frac{x_0}{E[(H - u^{-1}(\alpha))^+]} \text{ for some } a > 0$$

is very rarely satisfied. The first way of constructing \tilde{A} , which seems to be natural, is to find a constant \bar{a} such that

$$\bar{a} = \inf \left\{ a : \bar{Q}\left\{\frac{dP}{d\bar{Q}} \geq a(H - u^{-1}(\alpha))^+\right\} \leq \frac{x_0}{\mathbf{E}^{\bar{Q}}[(H - u^{-1}(\alpha))^+]} \right\}.$$

and then expect that

$$\bar{A} = \left\{ \frac{dP}{d\bar{Q}} \geq \bar{a}(H - u^{-1}(\alpha))^+ \right\}$$

is a solution. Unfortunately, this is not a right construction as shown in the example below.

Example

Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and P and Q are two measures given by $p_1 = \frac{7}{15}, p_2 = \frac{4}{15}, p_3 = \frac{4}{15}$ and $q_1 = \frac{4}{10}, q_2 = \frac{3}{10}, q_3 = \frac{3}{10}$. We want to maximize $P(A)$ subject to the condition $Q(A) \leq x_0 = \frac{6}{10}$. We have $\frac{p_1}{q_1} = \frac{63}{54}, \frac{p_2}{q_2} = \frac{48}{54}, \frac{p_3}{q_3} = \frac{48}{54}$ and the above construction gives $\tilde{A} = \{\omega_1\}$. However $Q(\{\omega_2, \omega_3\}) = \frac{6}{10}$ and $P(\{\omega_2, \omega_3\}) = \frac{8}{15} > \frac{7}{15} = P(\omega_1)$.

Below we present a lemma which provides construction of \tilde{A} when measures satisfy some particular condition. It turns out that this condition is satisfied by a significant number of cases in the hedging problem of call option.

Lemma 5.1 *Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ and measures P and Q (not necessary probabilistic) satisfy the following conditions:*

$$p_1 \geq p_2 \geq p_3 \geq \dots \geq p_n > 0 < q_1 \leq q_2 \leq q_3 \leq \dots \leq q_n$$

and γ be a fixed constant. Let $\tilde{A} = \{\omega_1, \omega_2, \dots, \omega_k\}$, where the number k is such that $Q(\omega_1, \omega_2, \dots, \omega_k) \leq \gamma$ and $Q(\omega_1, \omega_2, \dots, \omega_k, \omega_{k+1}) > \gamma$. Then $P(\tilde{A}) \geq P(A)$ for any set A satisfying $Q(A) \leq \gamma$.

Proof :

Let $B \subseteq \Omega$ s.t. $Q(B) \leq \gamma$.

1) First assume that $\tilde{A} \cap B = \emptyset$. Then $|B| \leq k$ and we have $P(\tilde{\omega}) \geq P(\omega)$ for each $\tilde{\omega} \in \tilde{A}$ and $\omega \in B$. As a consequence

$$P(\tilde{A}) = \sum_{\omega \in \tilde{A}} P(\omega) \geq \sum_{\omega \in B} P(\omega) = P(B).$$

2) If $\tilde{A} \cap B \neq \emptyset$ then by (1) $\tilde{A} \setminus \{\tilde{A} \cap B\}$ is a solution of

$$P(A) \longrightarrow \max$$

$$Q(A) \leq \gamma - Q(\{\tilde{A} \cap B\})$$

and so $P(\tilde{A} \setminus \{\tilde{A} \cap B\}) \geq P(B \setminus \{\tilde{A} \cap B\})$. As a consequence $P(\tilde{A}) \geq P(B)$. \square

Since $P(\omega_k)$ increases with k if $p > \frac{1}{2}$ and decreases if $p < \frac{1}{2}$, the only point to apply the lemma is to state the monotonicity of the measure \bar{Q} . In fact we are interested in monotonicity of \bar{Q} only on the set where it is strictly positive. Let us denote

$$a_k := \bar{Q}(\omega_k) = (S(1+u)^k(1+d)^{N-k} - \bar{K})^+ p^{*k} (1-p^*)^{N-k},$$

$$b_k := \frac{(S(1+u)^{k+1}(1+d)^{N-k-1} - \bar{K})^+}{(S(1+u)^k(1+d)^{N-k} - \bar{K})^+},$$

$$q := \frac{1+u}{1+d},$$

where the sequence b_k is well defined under convention that $\frac{a}{0} = \infty$ for $a \geq 0$. Then $\bar{Q}(\omega_k)$ is increasing if $\frac{a_{k+1}}{a_k} \geq 1$ for each $k = 0, 1, \dots, N-1$. This condition is equivalent to that $b_k \geq \frac{1-p^*}{p^*}$ for each $k = 0, 1, \dots, N-1$. But now note that the sequence b_k is decreasing. To see that one can calculate that

$$\frac{b_{k+1}}{b_k} \leq 1 \iff (q-1)^2 \geq 0.$$

The last condition is always satisfied. Thus $\bar{Q}(\omega_k)$ is increasing if

$$b_{N-1} = \frac{(S(1+u)^N - \bar{K})^+}{(S(1+u)^{N-1}(1+d) - \bar{K})^+} \geq \frac{1-p^*}{p^*}.$$

Note that this case includes the situations when $p^* \geq \frac{1}{2}$.

By analogous arguments one can obtain condition under which $\bar{Q}(\omega_k)$ is decreasing. This is the case when the $b_{\bar{k}} \leq \frac{1-p^*}{p^*}$, where \bar{k} is the minimal k for which $b_k \neq \infty$. Indeed, then we have $b_k \leq \frac{1-p^*}{p^*}$ for all $k \geq \bar{k}$ what implies that $a_{k+1} < a_k$ for $k \geq \bar{k}$.

Before summarizing the above consideration let us introduce the following notation

$$A_k := \{\omega \in \Omega \text{ s.t. the number of jumps upwards is equal to } k\}.$$

for the set containing all elements ω_k . The following lemma is a consequence of lemma 5.1.

Lemma 5.2

1) (P increasing, \bar{Q} decreasing)

Let $\bar{k} = \min\{k : b_k \neq \infty\}$. If $p \geq \frac{1}{2}$ and $b_{\bar{k}} \leq \frac{1-p^*}{p^*}$ then $\tilde{A} = A_N \cup A_{N-1} \cup \dots \cup A_{N-\bar{k}} \cup B_{N-\bar{k}-1}$, where the number \bar{k} is s.t. $\bar{Q}(A_N \cup A_{N-1} \cup \dots \cup A_{N-\bar{k}}) \leq x_0$ and $\bar{Q}(A_N \cup A_{N-1} \cup \dots \cup A_{N-\bar{k}-1}) > x_0$ and the set $B_{N-\bar{k}-1}$ contains maximal number of any elements from the set $A_{N-\bar{k}-1}$ such that $\bar{Q}(B_{N-\bar{k}-1}) \leq x_0 - \bar{Q}(A_N \cup A_{N-1} \cup \dots \cup A_{N-\bar{k}})$.

2) (P decreasing, \bar{Q} increasing)

If $p \leq \frac{1}{2}$ and $\frac{(S(1+u)^{N-\bar{K}})^+}{(S(1+u)^{N-1}(1+d) - \bar{K})^+} \geq \frac{1-p^*}{p^*}$ (for example when $p^* \geq \frac{1}{2}$) then $\tilde{A} = A_0 \cup A_1 \cup \dots \cup A_k \cup B_{k+1}$, where the number k is s.t. $\bar{Q}(A_0 \cup A_1 \cup \dots \cup A_k) \leq x_0$ and $\bar{Q}(A_0 \cup A_1 \cup \dots \cup A_{k+1}) > x_0$ and the set B_{k+1} contains maximal number of any elements from the set A_{k+1} such that $\bar{Q}(B_{k+1}) \leq x_0 - \bar{Q}(A_0 \cup A_1 \cup \dots \cup A_k)$.

Example

As an application of lemma 5.2 we study a risk minimizing problem for a call option with strike $K = 600$ in a 3-period model with parameters : $S_0 = 1000$, $u = 0, 1$, $d = -0, 2$, $p = \frac{1}{4}$. Price at time 0 of the option is $u_0 = \mathbf{E}^Q[(S_3 - 600)^+] = 398\frac{7}{27}$. Assume that we have only $x_0 = 150$ and $\alpha = 5$ is a level of acceptable losses measured by $u(x) = \sqrt{x}$. We denote by ω^{abc} , where $a, b, c \in \{u, d\}$ elementary events with interpretation of a, b, c as a history of the price process. For example ω^{udu} means the event where the price process moved up in the first and the third period and moved down in the second one. Since we can not hedge the original contingent claim $H = (S_3 - 600)^+$:

$$H(\omega^{uuu}) = 731, \quad H(\omega^{uud}) = H(\omega^{udu}) = H(\omega^{duu}) = 368,$$

$$H(\omega^{udd}) = H(\omega^{dud}) = H(\omega^{ddu}) = 104, \quad H(\omega^{ddd}) = 0,$$

we have to hedge $\tilde{H} = \mathbf{1}_{\tilde{A}}(S_3 - 625)^+$. Since $p = \frac{1}{4}$ and $p^* = \frac{2}{3}$, we can apply lemma 5.2(2) for construction of \tilde{A} . Below we present three possible right candidates for \tilde{H} .

$$\tilde{H}(\omega^{uuu}) = 0, \quad \tilde{H}(\omega^{ddd}) = 0, \quad \tilde{H}(\omega^{ddu}) = \tilde{H}(\omega^{dud}) = \tilde{H}(\omega^{udd}) = 79$$

$$\text{and } \left\{ \tilde{H}(\omega^{uud}) = \tilde{H}(\omega^{udu}) = 343, \quad \tilde{H}(\omega^{ddu}) = 0 \right\}$$

$$\text{or } \left\{ \tilde{H}(\omega^{uud}) = 0, \quad \tilde{H}(\omega^{udu}) = \tilde{H}(\omega^{ddu}) = 343 \right\}$$

$$\text{or } \left\{ \tilde{H}(\omega^{uud}) = 343, \quad \tilde{H}(\omega^{udu}) = 0, \quad \tilde{H}(\omega^{ddu}) = 343 \right\}$$

Moreover, $P(\tilde{A}) = \left(\frac{3}{4}\right)^3 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 \cdot 3 + \left(\frac{1}{4}\right)^2 \cdot \frac{3}{4} \cdot 2 = \frac{15}{16}$.

6 Incomplete markets

Now let us consider the case when the equivalent martingale measure is not unique. This means that the market is incomplete and not every contingent claim can be replicated. We preserve all assumptions from previous section. Recall that the wealth process $X_t^{x, \pi}$ is a supermartingale with respect to each martingale measure $Q \in \mathcal{Q}$. In this case theorem which describes optimal strategy is of the form:

Theorem 6.1 *Assume that there exists set \tilde{A} which is a solution of the problem:*

$$P(A) \longrightarrow \max \sup_{Q \in \mathcal{Q}} E^Q[\mathbf{1}_A(H - u^{-1}(\alpha))^+] \leq x_0.$$

Then the strategy which hedges the contingent claim $\mathbf{1}_{\tilde{A}}(H - u^{-1}(\alpha))^+$ is optimal.

Proof :

Let us consider an arbitrary admissible strategy (x, π) , where $x \leq x_0$. We will show that $P(u(H - X_T^{x, \pi})^+ \leq \alpha) \leq P(\tilde{A})$.

Notice, that for any $a, b, c \geq 0$ we have $(a - b)^+ \leq c \iff b \geq (a - c)^+$ and thus $u((H - X_T^{x, \pi})^+) \leq \alpha \iff X_T^{x, \pi} \geq (H - u^{-1}(\alpha))^+$. As a consequence for any $Q \in \mathcal{Q}$ we obtain

$$\begin{aligned} E^Q[\mathbf{1}_{\{u((H - X_T^{x, \pi})^+) \leq \alpha\}}(H - X_T^{x, \pi})^+] &\leq E^Q[\mathbf{1}_{\{u((H - X_T^{x, \pi})^+) \leq \alpha\}}X_T^{x, \pi}] \\ &\leq E^Q[X_T^{x, \pi}] \leq x \leq x_0, \end{aligned}$$

where the last but one inequality follows from the fact that $X_t^{x,\pi}$ is a Q supermartingale. Taking supremum over all martingale measures we have

$$\sup_{Q \in \mathcal{Q}} E^Q[\mathbf{1}_{\{u((H - X_T^{x,\pi})^+) \leq \alpha\}} (H - u^{-1}(\alpha))^+] \leq x_0.$$

From the definition of the set \tilde{A} we have $P(u(H - X_T^{x,\pi})^+ \leq \alpha) \leq P(\tilde{A})$.

Now let us consider the strategy $(\tilde{x}, \tilde{\pi})$ which hedges $\mathbf{1}_{\tilde{A}}(H - u^{-1}(\alpha))^+$. We have

$$\{u(H - X_T^{\tilde{x}, \tilde{\pi}})^+ \leq \alpha\} = \{X_T^{\tilde{x}, \tilde{\pi}} \geq (H - u^{-1}(\alpha))^+\} \supseteq \{X_T^{\tilde{x}, \tilde{\pi}} \geq \mathbf{1}_{\tilde{A}}(H - u^{-1}(\alpha))^+\} \supseteq \tilde{A}$$

and so $P(u(H - X_T^{\tilde{x}, \tilde{\pi}})^+ \leq \alpha) \geq P(\tilde{A})$. It follows that $(\tilde{x}, \tilde{\pi})$ is optimal and moreover we have $P(u(H - X_T^{\tilde{x}, \tilde{\pi}})^+ \leq \alpha) = P(\tilde{A})$. \square

The main problem which needs to be investigated is the existence of the set \tilde{A} . We are not in a position to prove a general existence result for \tilde{A} but we will show an example of trinomial model where it can be explicitly found.

Example - Trinomial model

Let us consider a one-step model where the stock price is given by

$$S_1 = S(1 + \xi), \text{ where } P(\xi = a) = p_1, P(\xi = b) = p_2, P(\xi = c) = p_3 \\ a > b > c, p_1, p_2, p_3 > 0, p_1 + p_2 + p_3 = 1$$

and where the interest rate is equal to 0. Here S is an initial price and S_1 is a price at time 1. To obtain the arbitrage-free model we assume that $a > 0$ and $c < 0$. Contingent claim is denoted by $H = (H_1, H_2, H_3) = (H(\omega_1), H(\omega_2), H(\omega_3))$.

First let us study the structure of the set of all martingale measures \mathcal{Q} . Each $Q \in \mathcal{Q}$ is a triplet $Q = (q_1, q_2, q_3)$ which is a solution of the system

$$\begin{cases} q_1 S_0(1 + a) + q_2 S_0(1 + b) + q_3 S_0(1 + c) = S_0 \\ q_1 + q_2 + q_3 = 1 \\ q_1, q_2, q_3 > 0. \end{cases}$$

By direct computation we obtain that such triplet can be parametrized by q_1 . Precisely speaking each martingale measure is of the form:

$$Q = \left(q_1, \frac{c - a}{b - c} q_1 + \frac{c}{c - b}, \frac{a - b}{b - c} q_1 + \frac{b}{b - c} \right), \text{ where } q_1 \in (\underline{q}, \bar{q}) := \left(0 \vee \frac{b}{b - a}, \frac{c}{c - a} \right).$$

That means that each $Q \in \mathcal{Q}$ can be represented by

$$Q = \alpha Q_1 + (1 - \alpha) Q_2, \text{ where } \alpha \in (0, 1) \text{ and} \\ Q_1 = \left(\underline{q}, \frac{c - a}{b - c} \underline{q} + \frac{c}{c - b}, \frac{a - b}{b - c} \underline{q} + \frac{b}{b - c} \right), \\ Q_2 = \left(\bar{q}, \frac{c - a}{b - c} \bar{q} + \frac{c}{c - b}, \frac{a - b}{b - c} \bar{q} + \frac{b}{b - c} \right).$$

Thus \mathcal{Q} is a convex set with two vertexes Q_1, Q_2 . Now notice, that for any $A \in \mathcal{F}$ we have

$$\sup_{Q \in \mathcal{Q}} E^Q[\mathbf{1}_A(H - u^{-1}(\alpha))^+] \leq x_0 \text{ if and only if} \\ E^{Q_1}[\mathbf{1}_A(H - u^{-1}(\alpha))^+] \leq x_0 \quad \text{and} \quad E^{Q_2}[\mathbf{1}_A(H - u^{-1}(\alpha))^+] \leq x_0,$$

so the constraints for \tilde{A} is reduced to two vertex measures. As a consequence we are looking for a set \tilde{A} which is a solution of the problem

$$P(A) \longrightarrow \max$$

$$\begin{cases} E^{Q_1}[\mathbf{1}_A(H - u^{-1}(\alpha))^+] \bar{Q}_1(A) \leq x_0 \\ E^{Q_2}[\mathbf{1}_A(H - u^{-1}(\alpha))^+] \bar{Q}_2(A) \leq x_0 \end{cases}.$$

Now let us make concrete calculations for the case when $b > 0$. Then $Q_1 = (0, \frac{c}{c-b}, \frac{b}{b-c})$, $Q_2 = (\frac{c}{c-a}, 0, \frac{a}{a-c})$. Let $\bar{H} := (H - u^{-1}(\alpha))^+$, $\bar{H}_i := (H_i - u^{-1}(\alpha))^+$. Our problem is of the form:

$$\begin{aligned} \mathbf{1}_{\omega_1}(A)p_1 + \mathbf{1}_{\omega_2}(A)p_2 + \mathbf{1}_{\omega_3}(A)p_3 &\longrightarrow \max \\ \mathbf{1}_{\omega_2}(A)\frac{c}{c-b}\bar{H}_2 + \mathbf{1}_{\omega_3}(A)\frac{b}{b-c}\bar{H}_3 &\leq x_0 \\ \mathbf{1}_{\omega_1}(A)\frac{c}{c-a}\bar{H}_1 + \mathbf{1}_{\omega_3}(A)\frac{a}{a-c}\bar{H}_3 &\leq x_0 \end{aligned}$$

Since we do not have a general method of solving, we will check all possibilities depending on S, a, b, c, H, u, α . We will denote by $L_1 := \frac{c}{c-b}\bar{H}_2 + \frac{b}{b-c}\bar{H}_3$ and by $L_2 := \frac{c}{c-a}\bar{H}_1 + \frac{a}{a-c}\bar{H}_3$. We have the following description of the set \tilde{A} .

1. If $L_1 \leq x_0$ and $L_2 \leq x_0$ then $\tilde{A} = \{\omega_1, \omega_3, \omega_3\}$.
2. If $\min\{\frac{c}{c-b}\bar{H}_2, \frac{b}{b-c}\bar{H}_3\} > x_0$ or $\min\{\frac{c}{c-a}\bar{H}_1, \frac{a}{a-c}\bar{H}_3\} > x_0$ then $\tilde{A} = \emptyset$.
3. If $L_1 \leq x_0$ and $L_2 > x_0$ and $\min\{\frac{c}{c-a}\bar{H}_1, \frac{a}{a-c}\bar{H}_3\} \leq x_0$ then if
 - (a) $\max\{\frac{c}{c-a}\bar{H}_1, \frac{a}{a-c}\bar{H}_3\} > x_0$ and if
 - i. $\frac{c}{c-a}\bar{H}_1 \geq \frac{a}{a-c}\bar{H}_3$ then $\tilde{A} = \{\omega_2, \omega_3\}$,
 - ii. $\frac{c}{c-a}\bar{H}_1 < \frac{a}{a-c}\bar{H}_3$ then $\tilde{A} = \{\omega_1, \omega_2\}$
 - (b) $\max\{\frac{c}{c-a}\bar{H}_1, \frac{a}{a-c}\bar{H}_3\} \leq x_0$ and if
 - i. $p_1 \geq p_3$ then $\tilde{A} = \{\omega_1, \omega_2\}$,
 - ii. $p_1 < p_3$ then $\tilde{A} = \{\omega_2, \omega_3\}$,
4. If $L_1 > x_0$ and $L_2 \leq x_0$ and $\min\{\frac{c}{c-b}\bar{H}_2, \frac{b}{b-c}\bar{H}_3\} \leq x_0$ then if
 - (a) $\max\{\frac{c}{c-b}\bar{H}_2, \frac{b}{b-c}\bar{H}_3\} > x_0$ and if
 - i. $\frac{c}{c-b}\bar{H}_2 \geq \frac{b}{b-c}\bar{H}_3$ then $\tilde{A} = \{\omega_1, \omega_3\}$,
 - ii. $\frac{c}{c-b}\bar{H}_2 < \frac{b}{b-c}\bar{H}_3$ then $\tilde{A} = \{\omega_1, \omega_2\}$
 - (b) $\max\{\frac{c}{c-b}\bar{H}_2, \frac{b}{b-c}\bar{H}_3\} \leq x_0$ and if
 - i. $p_2 \geq p_3$ then $\tilde{A} = \{\omega_1, \omega_2\}$,
 - ii. $p_2 < p_3$ then $\tilde{A} = \{\omega_1, \omega_3\}$,
5. If $L_1 > x_0$ and $L_2 > x_0$ and $\min\{\frac{c}{c-b}\bar{H}_2, \frac{b}{b-c}\bar{H}_3\} \leq x_0$ and $\min\{\frac{c}{c-a}\bar{H}_1, \frac{a}{a-c}\bar{H}_3\} \leq x_0$ then if
 - (a) $\max\{\frac{c}{c-b}\bar{H}_2, \frac{b}{b-c}\bar{H}_3\} > x_0$ and $\max\{\frac{c}{c-a}\bar{H}_1, \frac{a}{a-c}\bar{H}_3\} > x_0$ and if

- i. $\frac{c}{c-b}\bar{H}_2 \leq \frac{b}{b-c}\bar{H}_3$ and $\frac{c}{c-a}\bar{H}_1 \leq \frac{a}{a-c}\bar{H}_3$ then $\tilde{A} = \{\omega_1, \omega_2\}$
- ii. $\frac{c}{c-b}\bar{H}_2 \leq \frac{b}{b-c}\bar{H}_3$ and $\frac{c}{c-a}\bar{H}_1 > \frac{a}{a-c}\bar{H}_3$ then $\tilde{A} = \{\omega_2\}$
- iii. $\frac{c}{c-b}\bar{H}_2 > \frac{b}{b-c}\bar{H}_3$ and $\frac{c}{c-a}\bar{H}_1 \leq \frac{a}{a-c}\bar{H}_3$ then $\tilde{A} = \{\omega_1\}$
- iv. $\frac{c}{c-b}\bar{H}_2 > \frac{b}{b-c}\bar{H}_3$ and $\frac{c}{c-a}\bar{H}_1 > \frac{a}{a-c}\bar{H}_3$ then $\tilde{A} = \{\omega_3\}$
- (b) $\max\{\frac{c}{c-b}\bar{H}_2, \frac{b}{b-c}\bar{H}_3\} > x_0$ and $\max\{\frac{c}{c-a}\bar{H}_1, \frac{a}{a-c}\bar{H}_3\} \leq x_0$ and if
 - i. $\frac{c}{c-b}\bar{H}_2 \leq \frac{b}{b-c}\bar{H}_3$ then $\tilde{A} = \{\omega_1, \omega_2\}$
 - ii. $\frac{c}{c-b}\bar{H}_2 > \frac{b}{b-c}\bar{H}_3$ and if
 - A. $p_3 \geq p_1$ then $\tilde{A} = \{\omega_3\}$
 - B. $p_3 < p_1$ then $\tilde{A} = \{\omega_1\}$
- (c) $\max\{\frac{c}{c-b}\bar{H}_2, \frac{b}{b-c}\bar{H}_3\} \leq x_0$ and $\max\{\frac{c}{c-a}\bar{H}_1, \frac{a}{a-c}\bar{H}_3\} > x_0$ and if
 - i. $\frac{c}{c-a}\bar{H}_1 \leq \frac{a}{a-c}\bar{H}_3$ then $\tilde{A} = \{\omega_1, \omega_2\}$
 - ii. $\frac{c}{c-a}\bar{H}_1 > \frac{a}{a-c}\bar{H}_3$ and if
 - A. $p_2 \geq p_3$ then $\tilde{A} = \{\omega_2\}$
 - B. $p_2 < p_3$ then $\tilde{A} = \{\omega_3\}$
- (d) $\max\{\frac{c}{c-b}\bar{H}_2, \frac{b}{b-c}\bar{H}_3\} \leq x_0$ and $\max\{\frac{c}{c-a}\bar{H}_1, \frac{a}{a-c}\bar{H}_3\} \leq x_0$ and if
 - i. $p_1 + p_2 \geq p_3$ then $\tilde{A} = \{\omega_1, \omega_2\}$
 - ii. $p_1 + p_2 < p_3$ then $\tilde{A} = \{\omega_3\}$.

References

- [1] M. Baran, *Quantile hedging on markets with proportional transaction costs*, *Applications Mathematicae* (2003), 193-208,
- [2] J. Cvitanić, I. Karatzas *On dynamic measures of risk*, *Finance and Stochastics* 3 (1999), 451-482,
- [3] H. Föllmer, P. Leukert *Quantile Hedging*, *Finance and Stochastics* 3 (1999), 251-273,
- [4] H. Föllmer, P. Leukert *Efficient Hedging: Cost versus Shortfall Risk*, *Finance and Stochastics* 4 (2000), 117-146,
- [5] J. Jacod, A.N. Shiryaev *Local martingales and the fundamental asset pricing theorems in the discrete-time case*, *Finance and Stochastics* 2 (1998), 259-273
- [6] H. Pham *Dynamic L^p -hedging in discrete time under cone constraints*, *SIAM J. Control Optim.* 38 (2000), No.3 665-682.